AIAA 81-4069

Evolution of Discontinuity at a Disturbance Wave Head in a Radiating Gas

Radhe Shyam,* V. D. Sharma,† and V. V. Menon‡
Banaras Hindu University, Varanasi, India

Introduction

A NUMBER of problems relating to wave propagation through ideal fluids or fluids with internal relaxation, which are in a constant state at rest before the arrival of the wave front, have been considered previously (a short account is given by Becker[1] and Clarke and McChesney[2]). But much less interest has been shown in the propagation of waves through nonuniform atmospheres, the exceptions being the work of Srinivasan and Vincenti[3] and Clarke.[4-6]

The present Note applies the theory of characteristics to study the behavior of plane, cylindrical, or spherical wave heads propagating through a spatially uniform but time-varying flow of a radiating gas near the optically thin limit. Exact predictions of the true nonlinear progress of the flow variable gradients at the wave head are made and an expression for the time taken to shock formation is obtained; in addition, the effect of initial wave-front curvature on the growth and decay behavior of nonplanar waves is also discussed.

The equations which describe the one-dimensional planar ($j=0$), cylindrically ($j=1$), or spherically ($j=2$) symmetrical motion of a radiating gas near the optically thin limit can be written in the familiar form (see Penner and Olfe,[7] pp. 296, 297, and 385)

$$\rho_t+u\rho_x+\rho u_x+jx^{-1}\rho u=0 \tag{1}$$

$$\rho u_t+\rho u u_x+p_x=0 \tag{2}$$

$$p_t+up_x+\rho a^2(u_x+jx^{-1}u)+(\gamma-1)q=0 \tag{3}$$

where u is the gas velocity, p the pressure, ρ the density, γ the ratio of specific heats, $a=(\gamma p/\rho)^{1/2}$ the speed of sound, t the time, and x the single spatial coordinate being either axial in flows with planar geometry or radial in cylindrically and spherically symmetric flows. The quantity $q=4\bar{k}_p\sigma T^4$ is the rate of energy loss by the gas per unit volume through radiation, where $\bar{k}_p$ is the mean absorption coefficient depending on the density ρ and temperature T, and σ is the Stefan-Boltzmann constant. Letter subscripts denote partial differentiation unless stated otherwise.

The equation of state is taken to be of the form

$$p=\rho RT \tag{4}$$

where R is the gas constant.

Characteristics and the Flow Ahead

Equations (2) and (3) can be rewritten in the form

$$p_\alpha+\rho a u_\alpha+\{(\gamma-1)q+jx^{-1}\rho ua^2\}t_\alpha=0 \tag{5}$$

$$p_\beta-\rho a u_\beta+\{(\gamma-1)q+jx^{-1}\rho ua^2\}t_\beta=0 \tag{6}$$

where α and β are two characteristic parameters defined such that

$$(\mathrm{d}x/\mathrm{d}t)_\beta=x_\alpha/t_\alpha=u+a \tag{7}$$

$$(\mathrm{d}x/\mathrm{d}t)_\alpha=x_\beta/t_\beta=u-a \tag{8}$$

Transformation from the space-time (x, t) plane to the plane of the characteristic parameters (α, β) will be one to one if the Jacobian

$$J=x_\alpha t_\beta-x_\beta t_\alpha=2at_\alpha t_\beta \tag{9}$$

does not vanish anywhere. In obtaining Eq. (9) use has been made of Eqs. (7) and (8). The following results are noted for future use:

$$f_t+uf_x=J^{-1}a(f_\alpha t_\beta+f_\beta t_\alpha) \tag{10}$$

$$f_x=J^{-1}(f_\alpha t_\beta-f_\beta t_\alpha) \tag{11}$$

where f can be either p, ρ, or u, etc. The relations

$$t_\alpha=-J\beta_x,\quad t_\beta=J\alpha_x,\quad x_\alpha=J\beta_t,\quad x_\beta=-J\alpha_t$$

are all readily verified.

The unperturbed field ahead of the wave whose behavior is to be investigated is assumed to be spatially uniform. All x derivatives therefore vanish and Eqs. (1-3) give

$$\rho_0=\text{constant} \tag{12}$$

$$u_0=0 \tag{13}$$

$$p_{0t}=-(\gamma-1)q_0 \tag{14}$$

where the subscript 0 indicates a value in the region ahead of the wave front $\beta(x,t)=0$ or $x=x(t)$, where $x(t)$ denotes the position on the wave front at any time t. The perturbations of the state ahead are assumed to propagate through radiating gas behind the wave front $\beta=0$. Across the wave front $\beta=0$, the parameters p, ρ, and u are essentially continuous but the discontinuity in their derivatives is permitted. We infer that a and q will behave similarly and that they will have their subscript 0 values at the wave head. Furthermore, any derivative with respect to α is continuous; discontinuities can appear only in the β derivatives.

Behavior at the Wave Front

Differentiation of Eq. (5) with respect to β and of Eq. (6) with respect to α, followed by subtraction and evaluation at $\beta=0+$ gives

$$2\rho_0a_0u^+_{\alpha\beta}+(\rho_0a_0)_\alpha u^+_\beta+jx^{-1}\rho_0a_0^2t_{0\alpha}u^+_\beta + (\gamma-1)\{q^+_\beta t_{0\alpha}-q_{0\alpha}t^+_\beta\}=0 \tag{15}$$

Quantities with a subscript 0 are continuous across $\beta=0$, while those with a superscript + may be discontinuous at the wave head, so that their values just behind the wave front are taken in Eq. (15).

Equations (1) and (2) with the help of Eqs. (9-11) can be written into the following equivalent form

$$(a\rho_\alpha+\rho u_\alpha)t_\beta+(a\rho_\beta-\rho u_\beta)t_\alpha+2jx^{-1}\rho uat_\alpha t_\beta=0 \tag{16}$$

$$(\rho au_\alpha+p_\alpha)t_\beta+(\rho au_\beta-p_\beta)t_\alpha=0 \tag{17}$$

Received July 23, 1980.

*Research Fellow, Applied Mathematics Section, Institute of Technology.

†Lecturer, Applied Mathematics Section, Institute of Technology, presently, Post Doctoral Fellow, Dept. of Aerospace Engineering, University of Maryland.

‡Reader, Applied Mathematics Section, Institute of Technology.

which, on evaluation at $\beta=0+$ and using Eqs. (12-14), yield

$$a_0\rho_\beta^+=\rho_0 u_\beta^+ \tag{18}$$

$$p_\beta^+ t_{0\alpha}-p_{0\alpha}t_\beta^+=\rho_0 a_0 u_\beta^+ t_{0\alpha} \tag{19}$$

Further, in view of Eqs. (4), (11), (18), and (19), the expression $q=4\bar{k}_p\sigma T^4$ yields

$$q_\beta^+ t_{0\alpha}-q_{0\alpha}t_\beta^+=\frac{q_0}{a_0}\left[4(\gamma-1)+(\bar{k}_p)^{-1}\left\{T_0(\gamma-1)\left(\frac{\partial\bar{k}_p}{\partial T}\right)_o+\rho_0\left(\frac{\partial\bar{k}_p}{\partial\rho}\right)_o\right\}\right]u_\beta^+ t_{0\alpha} \tag{20}$$

Equations (15) and (20) yield the following equations for the variation of u_β^+ with α:

$$\frac{\partial}{\partial\alpha}\log\{(\rho_0 a_0)^{1/2}u_\beta^+\}=-\left\{\Lambda_1+\frac{ja_0}{2x}\right\}t_{0\alpha} \tag{21}$$

where

$$\Lambda_1=\frac{(\gamma-1)^2 q_0}{2\rho_0 a_0^2}\left[4+(\bar{k}_p)^{-1}\left\{T_0\left(\frac{\partial\bar{k}_p}{\partial T}\right)_o+\rho_0(\gamma-1)^{-1}\left(\frac{\partial\bar{k}_p}{\partial\rho}\right)_o\right\}\right]$$

Integrating Eq. (21) between α_i (where $u_\beta^+=u_{\beta i}^+$ and $t=t_i$) and α, we obtain

$$u_\beta^+=u_{\beta i}^+\left(\frac{\rho_{0i}a_{0i}}{\rho_0 a_0}\right)^{1/2}\exp\left\{-\int_{t_i}^t\left(\Lambda_1+\frac{ja_0}{2x}\right)\mathrm{d}t\right\} \tag{22}$$

In order to discover how u_x, say, is changing at $\beta=0+$ it is necessary to find t_β^+, since Eqs. (9), (11), and (13) give

$$u_x^+=-u_\beta^+/2a_0 t_\beta^+ \tag{23}$$

Equations (7) and (8) show that

$$2at_{\alpha\beta}+(u_\beta+a_\beta)t_\alpha-(u_\alpha-a_\alpha)t_\beta=0 \tag{24}$$

so that at $\beta=0+$

$$2a_0(t_\beta^+)_\alpha+u_\beta^+ t_{0\alpha}+(a_\beta^+ t_{0\alpha}+a_{0\alpha}t_\beta^+)=0 \tag{25}$$

Since $a^2=\gamma p/\rho$, it follows that

$$a_\beta t_\alpha+a_\alpha t_\beta=\frac{\gamma}{2\rho a}(p_\alpha t_\beta+p_\beta t_\alpha)-\frac{a}{2\rho}(\rho_\alpha t_\beta+\rho_\beta t_\alpha) \tag{26}$$

Combination of Eqs. (5) and (6) gives, on evaluation at $\beta=0+$,

$$p_{0\alpha}t_\beta^+ + p_\beta^+ t_{0\alpha}=\{\rho_0 a_0 u_\beta^+ - 2(\gamma-1)q_0 t_\beta^+\}t_{0\alpha} \tag{27}$$

where use has been made of Eq. (13).

Evaluation of Eq. (26) at $\beta=0+$, on using Eqs. (12), (14), (18), and (27), gives

$$a_\beta^+ t_{0\alpha}+a_{0\alpha}t_\beta^+=\frac{(\gamma-1)}{2}\left\{u_\beta^+-\frac{2\gamma q_0}{\rho_0 a_0}t_\beta^+\right\}t_{0\alpha} \tag{28}$$

Equations (25) and (28) yield

$$(t_\beta^+)_\alpha-\Lambda_2 t_{0\alpha}t_\beta^+ +\frac{(\gamma+1)}{4a_0}u_\beta^+ t_{0\alpha}=0 \tag{29}$$

where

$$\Lambda_2=\gamma(\gamma-1)q_0/2\rho_0 a_0^2$$

Equation (29), together with the result of Eq. (22), can be integrated to find the variation of t_β^+ with time at the wave head. Integration of Eq. (29), on using Eq. (22), yields

$$t_\beta^+=\exp\left(\int_{t_i}^t\Lambda_2(\hat{t})\mathrm{d}\hat{t}\right)\left[t_{\beta i}^+-u_{\beta i}^+\int_{t_i}^t\frac{(\gamma+1)}{2a_0}\left(\frac{\rho_{0i}a_{0i}}{\rho_0 a_0}\right)^{1/2}\times\exp\left\{-\int_{t_i}^{\hat{t}}\left(\mu_o+\frac{ja_0}{2x}\right)\mathrm{d}\tilde{t}\right\}\mathrm{d}\hat{t}\right] \tag{30}$$

where

$$\mu_o=\Lambda_1+\Lambda_2=\frac{(\gamma-1)q_0}{2\rho_0 a_0^2}\left[(5\gamma-4)+(\bar{k}_p)_0^{-1}\left\{T_0(\gamma-1)\left(\frac{\partial\bar{k}_p}{\partial T}\right)_0+\rho_0\left(\frac{\partial\bar{k}_p}{\partial\rho}\right)_0\right\}\right]$$

and $u_{\beta i}^+$ is the value of u_β^+ when $t=t_i$. Since $\bar{k}_p$ is an arbitrary function of ρ and T, the sign of μ_o may be positive or negative depending on the form of $\bar{k}_p$. [See, for example, 1) Ref. 7, p. 385, where $q\simeq ch^m\rho^k$, where h is the specific enthalpy given by $h=\gamma p/\rho(\gamma-1)$ and c, m, k are the constants approximately chosen for the regime of interest; 2) Ref. 7, p. 398, where $q=cT^d(\rho/\rho_0)^{nT^a}$, where ρ_0 is the reference density and c, d, n, and a are the constants approximately chosen for the regime of interest; and 3) Eq. (66) of Ref. 8.]

Equation (23) yields

$$u_{xi}^+=-u_{\beta i}^+/2a_{0i}t_{\beta i}^+ \tag{31}$$

Equation (23), together with Eqs. (22), (30), and (31), yields

$$u_x^+=\frac{u_{xi}^+(\rho_{0i}a_{0i}^3/\rho_0 a_0^3)^{1/2}\exp\left\{-\int_{t_i}^t\left(\mu_o+\frac{ja_0}{2x}\right)\mathrm{d}t\right\}}{1+\frac{(\gamma+1)}{2}u_{xi}^+\int_{t_i}^t\left[\left(\frac{\rho_{0i}a_{0i}^3}{\rho_0 a_0^3}\right)^{1/2}\exp\left\{-\int_{t_i}^{\hat{t}}\left(\mu_o+\frac{ja_0}{2x}\right)\mathrm{d}\tilde{t}\right\}\right]\mathrm{d}\hat{t}} \tag{32}$$

Equation (32) gives the variation of discontinuity in u_x^+ at $\beta=0$ which we have been seeking. It is clear from Eq. (32) that the temporal behavior of the velocity gradient at the wave head will depend critically on the sign of μ_o.

Discussion

If one considers only a short interval of time, so that a_0 and μ_o do not change appreciably between t_i and t, it is evident that Eq. (32) can be written in the approximate form

$$u_x^+\simeq\frac{u_{xi}^+(x/x_0)^{-j/2}\exp(-\bar{\mu}_o t)}{1+\frac{(\gamma+1)}{2}u_{xi}^+\int_0^t\left\{(x/x_0)^{-j/2}\exp(-\bar{\mu}_o\hat{t})\right\}\mathrm{d}\hat{t}} \tag{33}$$

where $\bar{a}_0$ and $\bar{\mu}_o$ indicate suitable mean values over the interval t_i to t, and t_i has been set equal to zero for convenience.

Moreover, $x = x_0 + \bar{a}_0 t$, x_0 being the value of x at $t=0$.

If $\bar{\mu}_o > 0$, an examination of Eq. (33) leads to the conclusion that for $u_{xi}^+ > 0$ (i.e., an expansive wave front), $u_x^+ \to 0$ as $t \to \infty$, i.e., the wave decays and damps out ultimately. If $u_{xi}^+ < 0$ (i.e., a compressive wave front), then there exists a positive initial critical value $(u_x^+)_c$ given by

$$(u_x^+)_c = \begin{cases} 2\bar{\mu}_o (\gamma+1)^{-1} & \text{for } j=0 \text{ (plane waves)} \\ \left\{ \left(\dfrac{\gamma+1}{2}\right) \displaystyle\int_0^\infty (x/x_0)^{-j/2} \exp(-\bar{\mu}_o t)\, dt \right\}^{-1} & \end{cases}$$

for cylindrical ($j=1$) and spherical ($j=2$) waves

such that for $|u_{xi}^+| < (u_x^+)_c$, $u_x^+ \to 0$ as $t \to \infty$ (i.e., the wave decays ultimately); for $|u_{xi}^+| = (u_x^+)_c$, $u_x^+ \to 2\bar{\mu}_o(\gamma+1)^{-1}$ as $t \to \infty$ (i.e., the wave takes a stable wave form); and for $|u_{xi}^+| > (u_x^+)_c$, $|u_x^+| \to \infty$ as $t \to t_c$ (i.e., the wave terminates in a shock wave in a finite time) where t_c is given by

$$t_c = (1/\bar{\mu}_o) \log\left\{1 + \frac{2\bar{\mu}_o}{(\gamma+1) u_{xi}^+}\right\}^{-1} \qquad \text{for } j=0$$

and

$$\int_0^{t_c} (x/x_0)^{-j/2} \exp(-\bar{\mu}_o t)\, dt = -2/\{(\gamma+1) u_{xi}^+\} \text{ for } j=1,2$$

One can see from the above expression of $(u_x^+)_c$ that $(\partial (u_x^+)_c / \partial \bar{\mu}_o) > 0$ and $(\partial (u_x^+)_c / \partial x_0) < 0$, which means that the thermal radiation (with $\bar{\mu}_o > 0$) and the wave-front curvature $(1/x_0)$ both have a stabilizing effect on the tendency of a wave head to grow into a shock in the sense that an increase in $\bar{\mu}_o$ or $(1/x_0)$ causes an increase in $(u_x^+)_c$.

If $\bar{\mu}_o < 0$, we note that the integral $\int_0^\infty (x/x_0)^{-j/2} \exp(-\bar{\mu}_o t)\, dt$ diverges to $+\infty$, and hence the initial critical value of the discontinuity vanishes. Thus Eq. (33) shows that for $\bar{\mu}_o < 0$, an expansive wave ultimately takes a stable wave form, i.e., for $u_{xi}^+ > 0$, $u_x^+ \to 2|\bar{\mu}_o|(\gamma+1)^{-1}$ as $t \to \infty$, while a compressive wave always steepens up into a shock in a finite time, no matter how small the initial discontinuity is, i.e., when $u_{xi}^+ < 0$, $|u_x^+| \to \infty$ as $t \to \bar{t}_c$, where $\bar{t}_c$ is given by

$$\bar{t}_c = (1/\bar{\mu}_o) \log\left\{1 + \frac{2|\bar{\mu}_o|}{(\gamma+1)|u_{xi}^+|}\right\} \text{ for } j=0$$

and

$$\int_0^{\bar{t}_c} (x/x_0)^{-j/2} \exp(|\bar{\mu}_o| t)\, dt = 2/\{(\gamma+1)|u_{xi}^+|\} \text{ for } j=1,2$$

In this case, $(\partial \bar{t}_c / \partial |\bar{\mu}_o|) < 0$ and $(\partial \bar{t}_c / \partial x_0) < 0$, which means that the thermal radiation (with $\bar{\mu}_o < 0$) causes a compressive wave to steepen into a shock more swiftly than it does in the absence of thermal radiation, whereas an increase in the initial wave-front curvature (x_0^{-1}) delays the shock formation.

References

[1]Becker, E., "Chemically Reacting Flows," *Annual Review of Fluid Mechanics*, Vol. 4, 1972, pp. 168-175.

[2]Clarke, J. F. and McChesney, M., *The Dynamics of Relaxing Gases*, Butterworths, London, 1976.

[3]Srinivasan, J. and Vincenti, W. G., "Criteria for Acoustic Instability in a Gas with Ambient Vibrational and Radiative Nonequilibrium," *Physics of Fluids*, Vol. 18, 1975, pp. 1670-1678.

[4]Clarke, J. F., "Diffusion Flame Stability," *Combustion Science and Technology*, Vol. 7, 1973, pp. 241-246.

[5]Clarke, J. F., "Behaviour at Acoustic Wave Fronts in a Laminar Diffusion Flame," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 27, 1974, pp. 161-173.

[6]Clarke, J. F., "Chemical Amplification at the Wave-Head of a Finite Amplitude Gasdynamic Disturbance," *Journal of Fluid Mechanics*, Vol. 81, 1977, pp. 257-264.

[7]Penner, S. S. and Olfe, D. B., *Radiation and Re-Entry*, Academic Press, New York and London, 1968.

[8]Pai, S. I., "Radiation Gasdynamics: High Temperature Gas Flow Including the Effects of Thermal Radiation," *Mecanique Appliquê*, Vol. 18, 1973, pp. 813-833.

AIAA 81-4070

Angular Distribution of Radiative Scattering: Comparison of Experiment and Theory

H. F. Nelson*
University of Missouri-Rolla, Rolla, Mo.

Introduction

MEASUREMENT of radiation scattered from suspended matter, which may consist of solid particles, liquid droplets or gas bubbles, is a standard diagnostic technique in the detection of sedimentation, combustion products, atmospheric fallout, flow fields in rocket nozzles, environmental pollution, aerosols, turbine blade lifetime, etc. These measurements are interpreted assuming single scattering and that Mie theory predicts the scattering phase function; consequently, one must be assured that Mie theory and experiment agree. The scattering of radiation by latex spheres has been found to agree roughly with Mie theory.[1] However, only a few investigations have studied the angular distribution of the scattered radiation over a wide range of angles.[2-8] These investigations have used both conventional sources[2-5] and laser sources[2,6-8] for the incident radiation, but the results were usually obtained for only one value of the particle size parameter[2-4,7], or the results emphasized polarization effects.[6,8] The objective of this Note is to report comparisons of Mie theory and experimental angular scattering of unpolarized radiation for a range of particle size parameters over a large angular range.

The present investigation uses polystyrene latex particles of very narrow size distribution immersed in distilled water to create the scattering medium. The uniform latex particles were obtained from the Dow Chemical Company, Lot Number: 3M4L in a 10% solution by volume. The particles had a mean diameter of 0.481 μm and a standard deviation of 0.0018 μm. The particle solutions were carefully diluted to particle volume concentrations of the order of 10^{-6} for the scattering measurements. A small volume of the emulsifier Triton X-100 was added to the solution to reduce the effects of aggregation.

A Brice-Phoenix light-scattering photometer was used to make the measurements at wavelengths of 0.436, 0.546, and 0.646 μm. The scattered radiation was measured at angles from 40 to 140 deg in 5 deg intervals. The measurements were made using a 26 mm diameter cylindrical cell with flat windows. It had a frosted inside back surface to reduce reflections. The standard narrow 4 mm apertures supplied by the

Received Aug. 15, 1980.

*Professor of Aerospace Engineering, Thermal Radiative Transfer Group, Dept. of Mechanical and Aerospace Engineering. Member AIAA.